# Modelling a Plant-Herbivore System with Simultaneous Effect of Disease and Toxicant

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**Abstract**

We propose a model to describe the interaction between plant population and herbivores when the plant population is diseased via vector population. Analysis of the system is performed to determine the stability of equilibrium points for a large range of parameter values. The solutions are shown to be uniformly bounded for all nonnegative initial conditions. The model predicts that in the absence of vector population, the infected plant population will not survive. Numerical simulation illustrates the dynamical behavior of the system.

**Keywords -** Plant Population, Vector Population, Herbivore, Disease, Toxicant, Stability.

## Introduction

Plant disease, although directly harmful to plants, also significantly influence the course of human events. Traditionally, in ecology, plant-herbivore interaction have been considered antagonistic i.e. herbivores have negative effect on plants. Thus both disease and herbivore have negative effect on plant population. Thousands of microbes and viruses cause infectious disease of plants. In contrast to pathogens of animals and humans, most plant pathogens are fungi, although bacteria, viruses and phytoplasms also causes problems compared with many other plant pathogens, the ecology and the evolution of plant viruses is particularly complex because most are transmitted between plants by vectors. Transmission of viruses by vectors is determined by many factors such as host susceptibility, the intensity of cropping, the number, mobility and life stage of the vectors, the spatial distribution and potency of virus-infected hosts and various other environmental and climatic conditions. There are relatively few studies that consider spatial processes when relating disease dynamics in the host plant population to the population dynamics of the vector (Holt *et al*., 1999; Ferriss *et al*., 1993; Ruesink *et al.*, 1986; Garrett *et al*., 1999). Recent advances in the development of mathematical models of plant-virus disease epidemics, which link vector population to host infection dynamics, have facilitated a consideration of the cause of virus disease epidemics and approaches to their alleviation (Nakasuji *et al*., 1885; Vandermeer *et al*.,1990; Holt *et al*., 1999; Jeger *et al*.,1998). The next threatening problem for the plant population is the effect of toxicant. Since long our environment is getting polluted by different types of chemicals, emitted due to various human activities, such as industrialization, use of pesticides and herbicides in agriculture etc. The biological and ecological consequences of a toxicant may be considered in several ways depending upon the toxic level and type of toxicants. The toxicant can affect the plant population in both direct and indirect ways. One of the visible direct or indirect effects is death of plant population. The direct effect of toxicant may also include alterations in direct mortality and reproductive rates. The indirect effects may be observed either through the food chain or through the reduction in the carrying capacity of the environment due to the degradation of habitat. Some mathematical models have been developed in this direction by several workers (Gordon *et al*., 1963; Hallam *et al*., 1984). In this paper, the simultaneous effect of both disease and toxicant will be studied in a plant-herbivore system.

**The Mathematical Formulation**

Let H(t) and S(t) be the healthy plant population and infected plant population. Z(t) be the non-healthy vector population and P(t) be the total vector population where and H1(t) be the herbivore population. Then the model 1(a) is given by the given set of ordinary differential equations:

$$\frac{dH}{dt} = r(K - H) - k_1 ZH - \theta 1 H H_1 \quad \dots (1)$$

$$\frac{dS}{dt} = k_1 ZH - (k3 + r)S - \theta 1 S H1 \quad \dots (2)$$

$$\frac{dZ}{dt} = k_2 S(P - Z) - cZ \quad \dots (3)$$

$$\frac{dH_1}{dt} = r_1(K)H1 - r_1(K)\frac{H_1^2}{H_0} - \alpha_4 H_1 S \quad \dots (4)$$

where r is the host mortality rate , K is the plant density, $\theta_1$ is the uptake rate of plant population by herbivore population, $(k_3 + r)$ is the sum of disease induced death rate and natural death rate of infected plant population, c is the natural death rate of vector population, $H_0$ is the carrying capacity of the herbivores, $r(K) = r_{30} + r_{33}K$ is the growth rate of herbivores depending on the plant density, $\alpha_4$ is the decay rate of herbivores due to interaction with infected plant population. Now, let C(t) is the Concentration of toxicant in the environment at time t, and U(t) is the concentration of toxicant in the organism at time t. Then the system (1-4) under the effect of toxicant can be given by the following system of ordinary differential equations 1(b)

$$\frac{dH}{dt} = r(K - H) - k_1 ZH - \theta 1 H H_1 - \alpha_1 HU \quad \dots (5)$$

$$\frac{dS}{dt} = k_1 ZH - (k3 + r)S - \theta 1 S H1 - \alpha_2 HU \quad \dots (6)$$

$$\frac{dZ}{dt} = k_2 S(P - Z) - cZ \quad \dots (7)$$

$$\frac{dH_1}{dt} = r_1(K)H1 - r_1(K)\frac{H_1^2}{H_0} - \alpha_4 H_1 S \quad \dots (8)$$

$$\frac{dC}{dt} = Q - hC \quad \dots (9)$$

$$\frac{dU}{dt} = a_1 C + \frac{d_1 \eta \varphi}{a_1} - (l_1 + l_2)U \quad \dots (10)$$

The first two terms on the right hand side in equation (10) denote the organismal net uptake of toxicant from the environment and the food chain, respectively; due to metabolic processing and other causes. The parameters $a_1$, $d_1$, $\varphi$, $l_1$, $\eta$ and $l_2$ are positive constants. $a_1$ denotes the environmental toxicant uptake rate per unit mass organism, d1 denotes the uptake rate of the toxicant in food per unit mass of organism, $\eta$ is the concentration of the toxicant in thr resource, $\varphi$, the average rate of food intake per unit mass organism, $l_1$ and $l_2$ are organismal net ingestion and depuration rates of toxicant respectively. The positive constant h in (10) represents the loss rate of toxicant from the environment including processes such as biological transformation, chemical hydolysis, volatization, microbial degradation and photosynthesis degradation. The exogenous rate of toxicant input into the environment is represented by Q.

Boundary Equilibria and Their Local Stability

In this section we will discuss local stability behavior of all the feasible boundary equilibrium points and the interior equilibrium point of the mathematical model (1)-(4) that we have discussed in previous section. System has two equilibrium points; the first equilibrium point is the disease-free equilibrium point i.e $E_1(H, S, Z, H_1)$ where $H = \frac{rK}{r + \theta_1 H_0}$ H=0,S=0,$H_1=H_0$.The second equilibrium point $E^*(H^*,S^*,Z^*,H_1^*)$ where $H_1^* = H_0(1 - \frac{\alpha_4 S^*}{r_1(K)})$ $Z^* = \frac{K_1 P S^*}{k_2 S^* + c}$,$H^* = \frac{rK}{r + k_1 Z^* + \theta_1 H_1^*}$ and $S^*$ is obtained from the cubic equation $AS^{*3} + BS^{*2} + CS^* + D = 0$

Where

$A = \theta_1 H_0 \alpha_4 a_1 a_3$

$B = \theta_1 H_0 \alpha_4 c a_3 - \theta_1 H_0 r_1(K) a_1 a_3 - \theta_1 H_0 \alpha_4 a_1 a_2 - \theta_1 H_0 \alpha_4 a_4$

$C = \theta_1 H_0 r_1(K a_1 a_2 + \theta_1 H_0 r_1(K) a_4 - \theta_1 H_0 \alpha_4 c a_2$

$D = \theta_1 H_0 r_1(K) c a_2 - rKr_{21}(K) k_1 k_2 P - \theta_1 H_0 r_1(K) c a_3$

$a_1 = k_2, a_2 \; r r_1(K) + \theta_1 H_0 r_1(K), a_3 = \theta_1 H_0 \alpha_4$

$a_4 = k_1 k_2 P r_1(K)$

Remark 1: Biologically it means that in the case of disease-free equilibrium point only the healthy plants are decreasing due to herbivores and the herbivore population is approaching to its carrying capacity as there is no negative impact on herbivores. But in the case of endemic equilibrium point, the healthy plant population goes to a very lower level because of the effect of both disease and herbivores. The jacobian corresponding to the system (1-4) gives the eigen values corresponding to the equilibrium points. At the equilibrium point E1 path is attracting in all the directions H, S, Z, and H1 with the eigen values $-(r+\theta_1 H_0)$, $-(k_3+r+\theta_1 H_0)$, -c and $-r_1(K)$, respectively. Hence it is locally asymptotically stable.

Now, we will discuss the linear stability of the system by lyapunov's direct method. Let us consider a positive definite function

$$V_1(H, S, Z, H_1) = \frac{n_1^2}{2} + \frac{n_2^2}{2} + \frac{n_3^2}{2} + \frac{n_4^2}{2} \qquad \text{... (11)}$$

then the time derivative of above equation is given as:

$$\dot{V}_1(t) = n_1 \dot{n}_1 + n_2 \dot{n}_2 + n_3 \dot{n}_3 + n_4 \dot{n}_4 \qquad \text{... (12)}$$

Where, $n_1 = (H - H^*)$, $n_2 = (S - S^*)$, $n_3 = (Z - Z^*)$, $n_4 = (H_1 - H^*)$
Now, from equations (1)-(4) and from equation (12) we get:

$$\dot{V}_1(t) = V_{11}^{'}(t) + V_{12}^{'}(t) + V_{13}^{'}(t) + V_{14}^{'}(t) \qquad \text{... (13)}$$

Where

$$V_{11}^{'}(t) = -((r + K_1 Z^* + \theta_1 H_1^*) n_1^2 + K_1 H^* n_1 n_3 + \theta_1 H_1^* n_1 n_4)$$

$$V_{12}^{'}(t) = -((K_3 + r + \theta_1 H_1^*) n_2^2 - K_1 H^* n_2 n_3 - K_1 Z^* n_1 n_2 + \theta_1 S^* n_2 n_4)$$

$$V_{13}^{'}(t) = -((K_2 S^* + c) n_3^2 + (K_2 Z^* - K_2 P) n_2 n_3)$$

$$V_{13}^{\cdot}(t) = -((\frac{r_1(K)H_1^*}{H_0} + \alpha_4 S^* - r_1(K))\, n_4^2 + \alpha_4 H_1^* n_1 n_4)$$

Then we can write

$$\dot{V}_1^* = a_{11}n_1^2 + a_{22}n_2^2 + a_{33}n_3^2 + a_{44}n_4^2 + a_{12}n_1n_2 + a_{13}n_1n_3 + a_{14}n_1n_4 + a_{23}n_2n_4 + a_{24}n_2n_4 \qquad \text{... (14)}$$

Where

$a_{11} = r + K_1Z^* + \theta_1H_1^*$, $a_{12} = -K_1Z^*$, $a_{13}$=$K_1H^*$, $a_{22} = K_3 + r + \theta_1H_1^*$,$a_{44} = \frac{r_1(K)H_1^*}{H_0} + \alpha_4S^* - r_1(K)$,
$a_{33} = K_2S^* + c$, $a_{23} = K_2Z^* - K_2P$,$a_{24} = \theta_1S^*$, $a_{14} = \theta_1H_1^*$.

Now, we see that by Sylvester's criteria under the following conditions

(i)$9a_{12}^2 < 4a_{11}a_{22}$, (ii)$3a_{13}^2 < 2a_{11}a_{33}$, (iii)$3a_{14}^2 < 2a_{11}a_{44}$, (iv)$3a_{24}^2 < 2a_{22}a_{44}$,(v) $3a_{23}^2 < 2a_{22}a_{33}$

Clearly, by Lyapunov's direct method E* is locally asymptotically stable.

Next we will discuss the boundedness of the system (1)-(4) in the following lemma.
Lemma 1: The system is mathematically posed in the region:

$$G = (\frac{rK}{\gamma} \le H + S \le K, 0 \le \frac{k_2KP}{C}, \frac{(r_{30} - \alpha_4K)H_0}{r_1(K)} \le H_1 \le H_0) \text{ where } \gamma = \max(r + \theta_1H_0, k_3 + r + \theta_1H_0)$$

Proof. From equation (1) and (2),we get:

$$\frac{d(H+S)}{dt} \le rK - rH - (k3 + r)S$$

Then from the above equation we get the following expression as $t \to \infty$:

$$\frac{d(H+S)}{dt} \le K$$

From equation (3), we get

$$\frac{dZ}{dt} \le k_2SP - k_1SZ - cZ$$

Then we get

$$\frac{dZ}{dt} \le k_2S_{max}P - cZ$$

then by usual comparison theorem [Hale, 1980], we get the following expression as $t \to \infty$:

$$\frac{dZ}{dt} \le \frac{k_2KP}{c}$$

From equation (4) we get the following expression as $t \to \infty$:

$$\frac{dH}{dt} \le H_0$$

Again, from equation (1) and (2), we get

$$\frac{d(H+S)}{dt} \ge rK - (r + \theta_1H_0)H - (k_3 + r + \theta_1H_0)S$$

Let we consider $\gamma = \max(r + \theta_1H_0, k_3 + r + \theta_1H_0)$ then

$$\frac{d(H+S)}{dt} \geq \frac{rK}{\gamma}$$

From equation (8), we get

$$\frac{dH_1}{dt} \geq AH_1 - BH_1^2$$

Where, $A = r_{30} - \alpha_4 S_{max}$ and $B = \frac{r_{1(K)}}{H_0}$

then by usual comparison theorem [Hale *et al.*,1980], we get the following expression as t →∞:

$$\frac{dH_1}{dt} \geq \frac{A}{B}$$

This completes the proof of lemma.

Now, we will study the global stability of the system in the region G in the following theorem again by Lyapunov direct method.

**Theorem 1:** E* is globally asymptotically stable if it satisfies the following conditions:

(i)$a_{24}^2 < 4a_{44}a_{22}$, (ii)$a_{23}^2 < 4a_{22}a_{33}$, (iii)$a_{14}^2 < 4a_{11}a_{44}$,

Where $a_{11} = r + kZ^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2}$ , $a_{22} = A_1(r + k_3 + \theta_1\frac{A}{B} - \frac{Kk_1}{2} - \frac{k^2KP}{c})$

$a_{33} = A_2(c + k_2S^* - \frac{Kk_1}{2} - \frac{k^2KP}{c})$ Where $a_{11} = \frac{r_1(K)H_1^*}{H_0} + \frac{r_1(K)A}{BH_0} + \frac{\alpha_4 rK}{\gamma} - r_1(K)$

$a_{14} = \theta_1H^*$, $a_{24} = \theta_1H^* + \alpha_4H_1^*$, $a_{23} = -k_2P$

**Proof:** . Let us consider a positive definite function

$$V_2(H,S,Z,H_1) = \frac{n_1^2}{2} + A_1\frac{n_2^2}{2} + A_2\frac{n_3^2}{2} + A_3\frac{n_4^2}{2} \quad \text{... (15)}$$

then the time derivative of above equation is given as:

$$\dot{V}_2(t) = n_1\dot{n}_1 + A_1n_2\dot{n}_2 + A_2n_3\dot{n}_3 + A_3n_4\dot{n}_4 \quad \text{... (16)}$$

Where, $n_1 = (H - H^*)$, $n_2 = (S - S^*)$, $n_3 = (Z - Z^*)$, $n4 = (H_1 - H_1^*)$

Now, from equations (1)-(4) and from equation (16) we get:

$$\dot{V}_2(t) = \dot{V_{21}}(t) + \dot{V_{22}}(t) + \dot{V_{23}}(t) + \dot{V_{24}}(t) \quad \text{... (17)}$$

Where

$$\dot{V_{21}}(t) = -(\left(r + K_1Z^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2}\right)n_1^2 + \frac{Kk_1}{2}n_2^2 - \frac{Kk_1}{2}n_3^2 + \theta_1H^*n_1n_4)$$

$$\dot{V_{22}}(t) = -(\left(K_3 + r + \theta_1\frac{A}{B} - \frac{Kk_1}{2}\right)A_1n_2^2 - A_1\frac{Kk_1}{2}n_3^2 - K_1Z^*A_1n_1n_2 + \theta_1S^*A_1n_2n_4)$$

$$\dot{V_{23}}(t) = -(\left(K_2S^*A_2 + cA_2 - \frac{A_2k^2KP}{c}\right)n_3^2 - \frac{A_2k^2KP}{c}n_2^2 - K_2PA_2n_2n_3)$$

$$\dot{V_{24}}(t) = -((\frac{r_1(K)H_1^*A_3}{H_0} + \frac{\alpha_4 rKA_3}{\gamma}S^* + \frac{A_3r_1(K)A}{BH_0} - r_1(K)A_3)\, n_4^2 + \alpha_4H_1^*A_3n_2n_4)$$

Then we can write

$$\dot{V}_1^* = -(a_{11}n_1^2 + a_{22}n_2^2 + a_{33}n_3^2 + a_{44}n_4^2 + a_{12}n_1n_2 + a_{14}n_1n_4 + a_{34}n_3n_4) \quad \text{... (18)}$$

Where

$a_{11} = r + K_1Z^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2}$, $a_{22} = A_1(K_3 + r + \theta_1\frac{A}{B} - \frac{Kk_1}{2})$,

 

$a_{44} = \frac{r_1(K)H_1^*A_3}{H_0} + \frac{\alpha_4 rKA_3}{\gamma}S^* + \frac{A_3 r_1(K)A}{BH_0} - r_1(K)A_3$ , $a_{33} = K_2S^*A_2 + cA_2 - \frac{A_2k^2KP}{c}$

$a_{23} = -K_2P$ , $a_{24} = \theta_1 S^* + \alpha_4 H_1^*$, $a_{14} = \theta_1 H_1^*$.

Now, we see that by Sylvester's criteria under the following conditions $V_2(t)$ is negative definite.

(i)$a_{24}^2 < 4a_{44}a_{22}$, (ii)$a_{23}^2 < 4a_{22}a_{33}$, (iii)$a_{14}^2 < 4a_{11}a_{44}$,

Clearly, by Lyapunov's direct method E* is globally asymptotically stable. This completes the proof of the theorem.

Now we will discuss the stability of the system (5-10). Again for the system, we will analyze the disease-free equilibrium point effected by toxicant that is, E1(H1, 0,X1, 0, P,U) and the endemic equilirium point in the polluted environment that is E*(H*, S*,X*,Z*, C*,U*). Now, consider the following system:

$$x'(t) = f(t, x) \qquad \text{... (19)}$$

$$y'(t) = g(y) \qquad \text{... (20)}$$

where, f and g are continuous and locally Lipschitz in x in Rn, and solutions exists for all positive time. Equation (20) is called asymptotically autonomous with limit equation (19) if f(t, x) → g(y) as t → ∞ uniformly for all x in $R^n$.

**Lemma 2 :** (see [13]) Let e be a locally asymptotically stable equilibrium of (20) and ω be the ω -limit set of a forward bounded solution x(t) of (19). If ω contains a point y0 such that the solutions of (20), with $y(0) = y_0$ converges to e as t →∞, then ω = {e} i.e. x(t) → e as t →∞.

**Corollary:** If the solutions of the system (19) are bounded and the equilibrium e of the limit system (20) is globally asymptotically stable than any solution x(t) of the system (19) satisfies x(t) → e as t → ∞. The equation (9) and (10) can be solved explicitly and we obtain

$$\lim_{t\to\infty} \sup C(t) \le C^* = \frac{Q}{h}$$

And

$$\lim_{t\to\infty} \sup U(t) \le U^* = \frac{a_1C^* + {}^{d_1\varphi\eta}/_{a_1}}{l_1 + l_2}$$

Thus, on applying above corollary in system (5)-(10) we get the following equivalent asymptotic autonomous system (Thieme *et al.*, 1992).

$$\frac{dH}{dt} = r(K - H) - k_1ZH - \theta 1HH_1 - \alpha_1 HU^* \qquad \text{... (21)}$$

$$\frac{dS}{dt} = k_1ZH - (k3 + r)S - \theta 1SH1 - \alpha_2 HU^* \qquad \text{... (22)}$$

$$\frac{dZ}{dt} = k_2S(P - Z) - cZ \qquad \text{... (23)}$$

$$\frac{dH_1}{dt} = r_1(K)H1 - r_1(K)\frac{H_1^2}{H_0} - \alpha_4 H_1 S \qquad \text{... (24)}$$

Thus, it is clear that the asymptotic behavior of the system (5)-(10) is equivalent to the asymptotic behaviour of the system (21)-(24), so that if the system (21)-(24) is stable then so is system (5)-(10). System (21)-(24) has also two equilibrium points, Firstly disease-free equilibrium point in the presence toxicant, that is, $E_1(H,S ,Z,H_1)$ where $H = \frac{rK}{r + \alpha_1 U^* + \theta_1 H_1}$ and $H_1=H_0$ secondly, endemic equilibrium point in polluted environment, that is, $E^*(H^*, S^*, X^*, Z^*)$ where $H_1 = H_0(1 - \frac{\alpha_4 S}{r_1(K)})$, $Z^* = \frac{k_2 S^* P}{c + k_2 S^*}$

$H^* = \dfrac{rK}{r + k_1 Z + \theta_1 H_1 + \alpha_1 U^*}$

And S* is obtained from $AS^{*3} + BS^{*2} + CS^* + D = 0$

$A = \theta_1 H_0 \alpha_4 a_1 a_3$
$B = \theta_1 H_0 \alpha_4 c a_3 - \theta_1 H_0 r_1(K) a_1 a_3 - \theta_1 H_0 \alpha_4 a_1 a_2 - \theta_1 H_0 \alpha_4 a_4$
$C = \theta_1 H_0 r_1(K) a_1 a_2 + \theta_1 H0 r_1(K) a_4 - \theta_1 H_0 \alpha_4 c a_2$
$D = \theta_1 H0 r_1(K) c a_2 - r_K r_{21}(K) k_1 k_2 P - \theta_1 H_0 r_1(K) c a_3$
$a_1 = k_2$, $a_2 = r r_1(K) + \theta_1 H_0 r_1(K) + r_1(K) \alpha_1 U^*$, $a_3 = \theta_1 H_0 \alpha_4$
$a_4 = k_1 k_2 P r_1(K)$

The Jacobian corresponding to the system (21-24) gives the eigen values corresponding to the equilibrium points. At the equilibrim point $E_1$ spath is attracting in all the directions H, S, Z, and $H^1$ with the Eigen values $-(r+ \theta_1 H_0 + \alpha_1 U^*)$, $-(k_3 + r + \theta_1 H_0 + \alpha_1 U^*)$, $-c$ and $-r_1(K)$, respectively. Hence it is locally asymptotically stable.

Now we will discuss the linear stability of the system by lyapunov's direct method. Let us consider a positive definite function:

$$V_3(H, S, Z, H_1) = \frac{n_1^2}{2} + \frac{n_2^2}{2} + \frac{n_3^2}{2} + \frac{n_4^2}{2} \qquad \text{... (25)}$$

then the time derivative of above equation is given as:

$$\dot{V}_3(t) = n_1 \dot{n}_1 + n_2 \dot{n}_2 + n_3 \dot{n}_3 + n_4 \dot{n}_4 \qquad \text{... (26)}$$

Where, $n_1 = (H - H^*)$, $n_2 = (S - S^*)$, $n_3 = (Z - Z^*)$, $n_4 = (H_1 - H_1^*)$

Now, from equations (21)-(24) and from equation (26) we get:

$$\dot{V}_3(t) = \dot{V}_{31}(t) + \dot{V}_{32}(t) + \dot{V}_{33}(t) + \dot{V}_{34}(t) \qquad \text{... (27)}$$

Where

$$\dot{V}_{31}(t) = -((r + K_1 Z^* + \theta_1 H_1^* + \alpha_1 U^*) n_1^2 + K_1 H^* n_1 n_3 + \theta_1 H_1^* n_1 n_4)$$

$$V_{32}(t) = -((K_3 + r + \theta_1 H_1^* + \alpha_2 U^*) n_2^2 - K_1 H^* n_2 n_3 - K_1 Z^* n_1 n_2 + \theta_1 S^* n_2 n_4)$$

$$\dot{V}_{33}(t) = -((K_2 S^* + c) n_3^2 + (K_2 Z^* - K_2 P) n_2 n_3)$$

$$\dot{V}_{34}(t) = -((\frac{2 r_1(K) H_1^*}{H_0} + \alpha_4 S^* - r_1(K))\, n_4^2 + \alpha_4 H_1^* n_1 n_4)$$

thus we can write equation (27) in the following form

$$\dot{V}_3^* = a_{11} n_1^2 + a_{22} n_2^2 + a_{33} n_3^2 + a_{44} n_4^2 + a_{12} n_1 n_2 + a_{13} n_1 n_3 + a_{14} n_1 n_4 + a_{23} n_2 n_4 + a_{24} n_2 n_4 \qquad \text{... (28)}$$

where

$a_{11} = r + K_1 Z^* + \theta_1 H_1^* + \alpha_1 U^*$, $a_{12} = -K_1 Z^*$, $a_{13} = K_1 H^*$, $a_{22} = K_3 + r + \theta_1 H_1^* + \alpha_2 U^*$, $a_{44} = \frac{2 r_1(K) H_1^*}{H_0} + \alpha_4 S^* - r_1(K)$, $a_{33} = K_2 S^* + c$, $a_{23} = K_2 Z^* - K_2 P - k_1 H^*$, $a_{24} = \theta_1 H^* + \alpha_4 H_1^*$, $a_{14} = \theta_1 H_1^*$.

Now, we see that by Sylvester's criteria under the following conditions $V_3(t)$ is negative definite.

(i) $9a_{12}^2 < 4a_{11}a_{22}$, (ii) $3a_{13}^2 < 2a_{11}a_{33}$, (iii) $3a_{14}^2 < 2a_{11}a_{44}$, (iv) $3a_{24}^2 < 2a_{22}a_{44}$, (v) $3a_{23}^2 < 2a_{22}a_{33}$

 

Clearly, by Lyapunov's direct method E* is locally asymptotically stable. In the next theorem we will prove the global stability of the system by Lyapunov's direct method.

**Theorem 2:** E* is globally asymptotically stable if it satisfies the following conditions:

(i) $3a_{12}^2 < 2a_{11}a_{22}$ (ii) $a_{14}^2 < a_{11}a_{44}$ (iii) $3a_{24}^2 < 4a_{22}a_{44}$, (iv) $3a_{23}^2 < 2a_{22}a_{33}$

Where

$a_{11} = r + K_1Z^* + \theta_1H_1^* + \alpha_1U^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2}$, $a_{12} = -K_1Z^*$, $a_{13} = K_1H^*$, $a_{22} = K_3 + r + \theta_1H_1^* + \alpha_2U^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2} + \frac{\alpha_4H_0}{2} - \frac{k^2KP}{2c}$, $a_{44} = \frac{r_1(K)H_1^*}{H_0} + \alpha_4S^* + + \frac{\alpha_4H_0}{2} + \frac{r_1(K)A}{BH_0} - r_1(K)$, $a_{33} = K_2S^* + c - \frac{k^2KP}{2c} - \frac{2Kk_1}{2}$, $a_{23} = -K_2P$, $a_{24} = \theta_1S^*$, $a_{14} = \theta_1H^*$.

**Proof:** Let us consider a positive definite function

$$V_4(H, S, Z, H_1) = \frac{n_1^2}{2} + B_1\frac{n_2^2}{2} + B_2\frac{n_3^2}{2} + B_3\frac{n_4^2}{2} \quad \text{... (29)}$$

then the time derivative of above equation is given as:

$$\dot{V}_4(t) = n_1\dot{n}_1 + B_1n_2\dot{n}_2 + B_2n_3\dot{n}_3 + B_3n_4\dot{n}_4 \quad \text{... (30)}$$

Where, $n_1 = (H - H^*)$, $n_2 = (S - S^*)$, $n_3 = (Z - Z^*)$, $n_4 = (H_1 - H_1^*)$

Now, from equations (21)-(24) and from equation (30) we get:

$$\dot{V}_4(t) = \dot{V}_{41}(t) + \dot{V}_{42}(t) + \dot{V}_{43}(t) + \dot{V}_{44}(t) \quad \text{... (31)}$$

Where

$$\dot{V}_{41}(t) = -(\left(r + K_1Z^* + \theta_1H_1^* + \alpha_1U^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2}\right)n_1^2 - \frac{Kk_1}{2}n_3^2 + \theta_1H^*n_1n_4)$$

$$\dot{V}_{42}(t) = -(\left(K_3 + r + \theta_1H_1^* + \alpha_2U^* + \theta_1\frac{A}{B} - \frac{Kk_1}{2} + \frac{\alpha_4H_0}{2} - \frac{k^2KP}{2c}\right)B_1n_2^2 + B_1\theta_1S^*n_4n_2 - B_1\frac{Kk_1}{2}n_3^2 - K_1Z^*B_1n_1n_2 + \theta_1S^*A_1n_2n_4)$$

$$\dot{V}_{43}(t) = -(\left(K_2S^* + c - \frac{k^2KP}{2c}\right)B_2n_3^2 - \frac{B_2k^2KP}{c}n_2^2 - K_2PB_2n_2n_3)$$

$$\dot{V}_{44}(t) = -((\frac{r_1(K)H_1^*}{H_0} + \alpha_4S^* + + \frac{\alpha_4H_0}{2} + \frac{r_1(K)A}{BH_0} - r_1(K))\,B_3n_4^2 + \frac{\alpha_4H_0n_2^2}{2})$$

Then we can write

$$\dot{V}_4 = -(a_{11}n_1^2 + a_{22}n_2^2 + a_{33}n_3^2 + a_{44}n_4^2 + a_{12}n_1n_2 + a_{14}n_1n_4 + a_{24}n_2n_4 + a_{23}n_2n_3) \quad \text{... (32)}$$

Now, we see that by Sylvester's criteria under the following conditions $V_4(t)$ is negative definite,

(i) $3a_{12}^2 < 2a_{11}a_{22}$ (ii) $a_{14}^2 < a_{11}a_{44}$ (iii) $3a_{24}^2 < 4a_{22}a_{44}$, (iv) $3a_{23}^2 < 2a_{22}a_{33}$

## Conclusion

In this paper, two models have been discussed. In the first model 1(a), the affect of disease on a plant. Herbivore system via vector population has been studied. It has been observed that both the plant population and herbivore population, decreases to a lower level as the disease affects it, but the plant population goes to a lower level due to the effect of both disease and herbivore (fig 1). In the second model 1(b), the affect of disease and toxicant has been studied on the plant population and it has been concluded that the plant population declines to a very lower level due to the simultaneous effect of both disease and toxicant (fig 2). Further, it has been observed that the vector population and the herbivore population also decrease to a lower level due to the effect of toxicant. The local and global stability conditions of both the models have been derived. The graphs of both the herbivore and healthy plant population have been plotted using MATLAB software. Numerical simulation of the system has been done in support of our result.

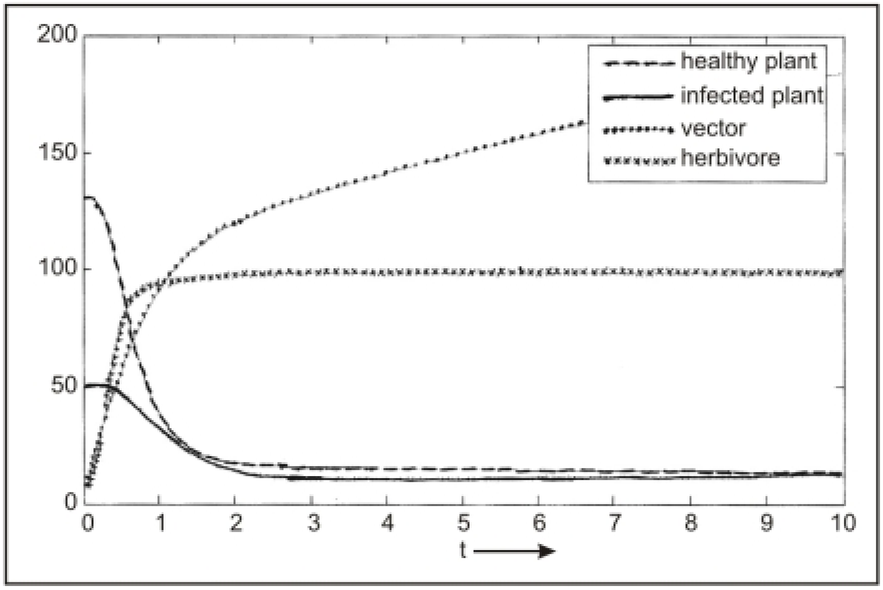


**Figure 1:** Healthy plants and herbivore effected by disease

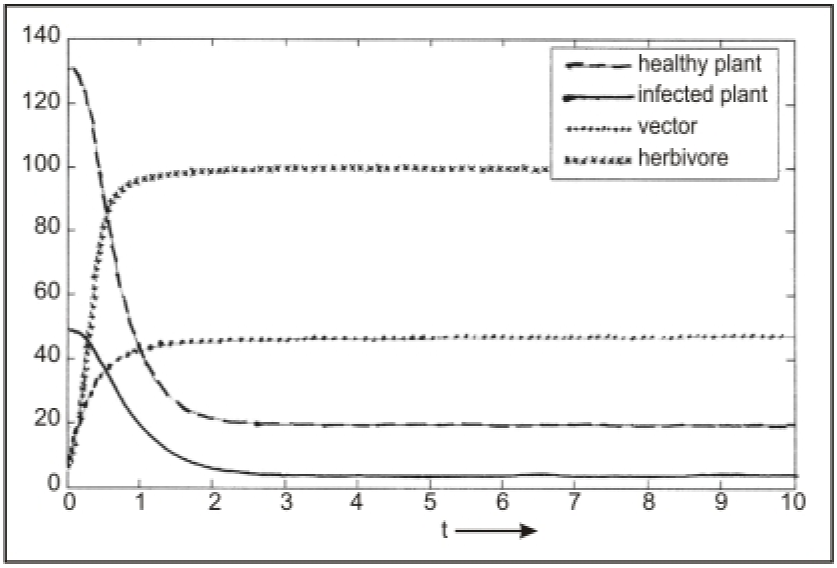


**Figure 2:** Effect of disease and toxicant on healthy plant population

 

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